# ENCAL: A Prototype Computer - Based Learning Environment for Teaching Calculator Representations

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## Extended Abstract

### *Pedagogy Required for Teaching/learning Arithmetic Representations*

A fundamental aspect of learning arithmetic is understanding and becoming proficient in abstract representational systems which convey concepts, such as algebra. A way of helping learners understand, particularly in the primary school, is to make connections between familiar concrete situations and more abstract symbolic concepts. This may be achieved through the use of objects which can be manipulated (e.g. base-10 blocks, sticks, counters, and computer graphics). Piagetian analysis indicates that for young children, the use of concrete manipulatives is important for the eventual development of formal operations (Shuard, et al, 1991). Children around 7-12 years of age have the ability to think logically (i.e. like adults) if their thinking is guided by contact with real or familiar objects and situations. The physical activity of say moving blocks, eventually leads to similar actions being carried out entirely in the imagination, and so at around 11-12 years of age, mental activities come to dominate and take the form of mental images which are moved around in the mind. It is the construction of mental images which are fundamental to the development of logical thinking and thinking in terms of concepts.

Mayer and Wittrock (1996) refer to this learning process as being structure-based, meaningful, and active. Meaningful learning is particularly appropriate to abstract concept acquisition, since this is: (a) concerned more with understanding than just a change in procedures; and (b) influenced by situations and domains as opposed to being independent from them (Shuell, 1992). A related theory is Constructivism, which asserts that both the active involvement in a situation by individuals, and the situation itself, affect cognitive growth (i.e. learning). Such constructive activity is based on Piaget's notion of assimilation and accommodation. That is, new procedures may be assimilated prior to learning, but meaningful learning (i.e. the understanding of concepts) occurs only after the accommodation of new schemata (Steffe, 1988). Thus, a new concept may be added to concepts already formed (i.e. constructed) as long as the new concept is accommodated to the existing concept network through understanding. If understanding does not take place, then an individual will simply assimilate new information and cognitive growth will not occur.

Cognitive psychology has grounded this analysis in the concept of mental models. For example, the theory of language understanding put forward by Johnson-Laird (1983) suggests that children construct concrete mental models which correspond to the entities (e.g. people, objects, events) which the language is about (as someone would do when listening to a story). A child then manipulates and mentally transforms the mental model, and this enables inferences concerning the language to be drawn.

Greeno (1991a) proposed an idea of mental models related to number sense similar to Johnson-Laird's (1983) language theory. Greeno (1991a) suggests that number sense is a form of cognitive expertise - in other words, it is the ability of a person to construct and reason with mental models. Thus, understanding the language of mathematics (e.g. a word problem) depends on learners developing the ability to construct mathematical *situations* which include the concepts that the language is putting forward. Greeno (1991a) refers to this as *situated cognition*, where the underlying assumptions concerning learning are: (a) the capabilities which people have with regard to number sense involve more than just facts and procedures; and (b) the activity of understanding and reasoning ultimately becomes internal (i.e. implicit) through the use of mental models.

The mental model theories above imply that when confronted with an abstract proposition, children need not think logically because they construct mental models of a situation by relating a proposition to the concrete or real world. Furthermore, since communication necessitates the use of external representations (e.g. with objects, symbols, and language), it may be assumed that: (a) internal representations (mental models) are influenced and constrained by external situations - situated cognition; and (b) connections between internal representations may be achieved as a result of external activity, thus facilitating the construction of knowledge networks (Hiebert and Carpenter, 1992). Consequently, there should be decreased dependency on the use of physical aids (e.g. arithmetic blocks) to facilitate thinking. Thus, when attempting to solve a problem in the conceptual domain of numbers, a person's reasoning will be guided by his/her internalised mental models formed through previous interactions of external concrete situations.

A fundamental aspect associated with both internal mental states of mind and external concrete situations is the pedagogical considerations of representation. These are addressed next.

### ***Pedagogical Considerations of Representation***

*Cognitive Science and the Connection of Representations*

Understanding mathematics is associated with the way information is presented and structured. The presentation of information should enable connections between ideas, facts, and procedures to be made. Cognitive science suggests that subsequent understanding will occur once a mental representation of a particular mathematical concept has become linked to a person's existing network of representations (i.e. from a psychological perspective, information has been accommodated).

With regard to representations and understanding, Hiebert and Carpenter (1992) build on two assumptions from cognitive science research by suggesting the following. Firstly, an internal representation is influenced by the represented external situation. Thus, connections between internal representations are influenced by connections between corresponding external representations, and so external mathematical representations influence internal mathematical representations. Secondly, internal representations can be connected. Similarly, Anderson (1993) states that the mind is a reflection of the external environment. However, Zhang (1997) goes further by arguing that external representation based problem solving (e.g. arithmetic multiplication) is constrained both by the environment and the mind of an individual. The fundamental assumption being that external representations do not have to be re-represented as internal representations in order for problem solving to be carried out. Zhang suggests that external representations can activate perceptual operations and provide perceptual information which may be used in conjunction with cognitive operations, such as existing internal representations (e.g. information recalled from memory).

Such theories are particularly useful when considering the design of learning environments in the field of arithmetic problem solving. Goldin (1987) describes the goal of mathematics education as being able to foster the development of cognitive (i.e. internal) representational systems. Consequently, Goldin points out that a teaching system needs to foster maximal development of pupils' internal representations using external representations which facilitate transfer of learning. The above assumptions from cognitive science suggest that having the ability to select an action in one external representation which can then be translated to other connected external representations would be a powerful pedagogical tool for activating and providing perceptual information and also representing ideas mentally. In this respect, the computer is extremely useful, because several different external representations can both be linked and structurally equivalent in order to facilitate the translation of actions carried out by pupils.

The distinction between internal and external representations helps understand why a particular technology may be better suited to the teaching and learning of arithmetic representations which involve the use of a calculator. Internal representations are the mental images people formulate in their minds which correspond to reality, whereas external representations are actual commodities which people can physically see and/or manipulate to depict reality, such as: symbols (e.g. algebra, diagrams, pictures); and real objects (e.g. arithmetic blocks, Cuisenaire rods, etc.). A representation may therefore be considered as being comprised of the three components: mental images, symbols, and real objects (Janvier, 1987).

Several different (i.e. multiple) types of external representation may be used to depict the same abstract concept, which in turn could help to promote understanding. For example, different types of external representation typically used in classrooms include: text books; writing/diagrams on white/black boards; and manipulable objects. However, Dufour-Janvier, et al. (1987) point that multiple external representations will only be useful if a child understands them, and where this occurs a learner will also be expected to: (a) find one representation which best enables a given task to be completed; (b) reject a representation because it is less effective than others in a given problem situation; and (c) move from one representation to another.

The movement between representations is problematic when considering how a solution is arrived at from a given problem statement. For example, Lesh, et al. (1987) found that pupils have “translation” (p. 36) difficulties associated with the re-representation of initial word problem information into ways of describing, illustrating, and manipulating ideas which may then be used for the solving of a problem. They point out that these difficulties arise not only within the context of word problems, but also with the translation to subsequent pencil and paper computations. Both the translation of information from word problem statements, and the subsequent translation of information to computations, are seen as significant factors in influencing mathematical learning. The translation difficulties highlighted by Lesh et al. will by implication also be apparent when considering the use of abstract calculator representations to facilitate computations.

The processes of mathematical thinking required to overcome such problems are based on complex relationships between the external representations encountered during learning, and internal mental processes (De Corte, et al. (1996). A theoretical model which classifies the unobservable thinking behaviour (i.e. internal representations) taking place in individuals is Goldin’s (1992b) model of internal representational systems. The model depicts complex processes of interaction between the following five internal representational systems: verbal/syntactic; heuristic (e.g. planning, and monitoring); formal notation (i.e. symbolic); affective; and imagistic (i.e. visual, spatial, auditory, tactile).

Conventional teaching tends to place an emphasis on verbally-mediated thinking during mathematics teaching (e.g. through explanations), and the use of formal types of notation (e.g. algebra). Consequently, De Corte, et al. (1996) point out that Goldin’s (1992b) model of mathematical thinking goes far beyond the presumed influence of verbal and formally written mathematics. Goldin's model implies that mathematical understanding (and thus teaching) needs to address the influence on learning of non-verbal imagery (i.e. visually mediated thinking), as well as the use of verbal and formal notational systems. This implication supports Paivio's (1986) dual-coding theory.

Having addressed relevant issues of learning and representation, the following section considers the pedagogical problems of teaching and learning using calculator representations.

### *Pedagogical Problems With Using Calculator Representations for Teaching/Learning Arithmetic*

Previous research asserts that the construction of concepts should precede the use of skills (Kaput, 1987; Hiebert, 1988). However, calculator usage typically promotes skills before arithmetic understanding. This is evident when one considers the theories underlying: meaningful learning, constructivism, mental models, and internal/external representations, all of which suggest that calculators will be of little educational value unless pupils understand the formal/abstract representations used.

Children, particularly at primary school level, cannot easily relate abstract data on a calculator display to concrete or real-life situations, and this will serve to impede mental model formation and thus learning. More specifically, because internal representations are influenced and constrained by external situations, and since the external representations of calculators are themselves abstract, it may be assumed that calculators constrain the development of internal representations and thus conceptual networks. In addition, the symbols viewed on a calculator display refer to abstract entities which are likely to be absent from pupils' cognitive structures (Greeno, 1991), and if this is the case, new information may be assimilated but not accommodated. Thus, if learning using calculators is to take place, it is important that number concepts are represented internally in a way which promotes understanding.

Unfortunately, conventional calculator representations do not lend themselves to mental model formation and thus conceptual or procedural understanding, for several reasons. Firstly, although calculators look concrete, they do not give perceptual representations to underlying abstractions (e.g. mappings between calculation steps, and evaluation sequences). In other words, data which is input remains abstract (i.e. in a symbolised format) which itself is unlikely to facilitate accommodation of knowledge and conceptual understanding. Secondly, calculators can cause confusion about procedural understanding (such as order of operations) due to the logic systems used to implement the calculation. Thirdly, calculators do not show intermediate stages of computations which could serve to support abstract understanding, and thus help with the construction of mental models and the accommodation of knowledge networks. For example, the reading of a word problem to the entering of data into a calculator is probably too large a step for the understanding of: (a) the entities involved, (b) the relationships between the entities, and (c) the evaluation sequence of an arithmetic expression. Fourthly, calculators do not facilitate planning, in particular the editing and reorganisation of data.

The potential pedagogical difficulties with calculator usage affect both teaching and learning, and so the next section considers a computer-based pedagogy which will be needed to overcome such problems.

### *A Computer-Based Pedagogy for Teaching Calculator Representations*

Once appropriate information has been elicited from the semantics of a word problem, what is of concern are the difficulties school pupils encounter with: (a) the translation of a problem statement to abstract arithmetic notation; (b) the manipulation of arithmetic notation for evaluation purposes; and (c) the understanding of calculator behaviour. Although calculators look concrete, they do not give perceptual representations to underlying abstractions (e.g. mappings between calculation steps, and evaluation sequences). In addition, calculators do not show intermediate stages of computations which could serve to support the conceptual and procedural understanding of arithmetic. In order to correctly compute an arithmetic function using a calculator which results from a word problem, it is often first necessary to: (a) identify entities and the relationships between them; (b) translate the entities to arithmetic notation, and vice versa; and (c) carry out computations through key presses. Consequently, the underlying question which has been addressed is: what pedagogical requirements will be needed to facilitate the teaching and understanding of abstract calculator representations when used for word problem solving? The main requirement is the use of multiple, equivalent, and linked representations (MELRs). Three structurally different, but equivalent computer-based representations have been used in the current design. These are: (i) iconic (concrete); (ii) dataflow (intermediate); and (iii) calculator (abstract).

#### *Iconic Representation*

Real-world icons are used to represent entities from problem statements and the corresponding arithmetic expressions in a more concrete and familiar way. This has been done because psychological theory suggests that individuals construct concrete mental models of entities, and then elaborate these models by manipulating and mentally transforming them (Johnson-Laird, 1983). In other words, a problem statement is related to the real world to facilitate understanding in the minds of children. For example, moving book icons to shelf icons simulates the manipulation of objects (i.e. books to shelves) in the real world. This direct manipulation of computer graphics serves three fundamental purposes during learning: (a) it acts as a spatial metaphor; (b) it enables problem entities to be concretised in the minds of individuals; and (c) it enhances the understanding of abstract concepts.

*Dataflow Representation*

The dataflow representation facilitates conceptual and procedural understanding between the concrete (iconic) and the abstract (calculator) representations. This is because the dataflow is an intermediate representation which serves as a pedagogic link due to the fact that it is neither wholly concrete nor wholly abstract. The intermediate dataflow representation is designed to help a user make the translation from the problem entities towards evaluation, and if necessary from evaluation back to the problem much easier, by: (a) providing a conceptual bridge (or pedagogic link) between the concrete icons, and the abstract symbols which will be used during calculator data entry; and (b) enabling users to carry out arithmetic manipulations prior to using the calculator for evaluations.

*Calculator Representation*

This is the most abstract of the three representations, and enables the formal system of mathematics notation to be depicted using a calculator. The calculator may be used either in arithmetic (four function) or scientific (algebraic) format. The calculator syntax can be compared concurrently with the equivalent dataflow and iconic representations to facilitate the understanding of calculation procedures. In addition, the behaviour of the calculator in terms of calculation sequence and answers to individual calculation steps, in either the arithmetic or scientific mode, is displayed and recorded next to the calculator. Users may then return to particular calculation steps to scrutinise possible sources of confusion with a calculation sequence.

The design of a prototype system (Entities, Notation, Calculator - ENCAL) is the current main focus of the research.

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