A Study about Students' Knowledge of Inductive Structures

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Abstract. This article describes two stages of a study carried out with pre-university students, to gather information about the learning of the concept of inductive structures. The study complements two previous investigations focusing on the design of recursive algorithms, from which the study of students' understanding about the input structures of the algorithms arises as a necessity. The theoretical framework used in the three studies is the epistemology of Jean Piaget, specially works about recursive reasoning on the series of natural numbers. Our methodology of research follows principles of Piaget's experiments in which the clinical method from psychiatry was adopted. In this sense, the instructional instance is a tool for obtaining information about cognitive processes. In the first stage, two instructional instances with eight voluntary participants were conducted, in which a problem about an inductively defined set is presented and some questions are posed. The analysis of the responses of the students reveals some difficulties casting doubts on students' conceptual knowledge on the series of natural numbers. Investigating this point is the goal of the second stage where one instructional instance is conducted with seven students, and new information is gathered and analyzed. The results of current and previous studies will be used to elaborate didactic material to introduce inductive definitions, recursive algorithms and proof by induction at pre-university level. This article describes the main theoretical guidelines, the development of both stages of the study and the analysis of the difficulties and progress observed. Some conclusions and future work are included.

Keywords: Data structures, induction-recursion, constructivism.

1 Introduction

In [11] Piaget and colleagues give empirical evidence about the psychogenetic evolution of mental structures[1] corresponding to *reasoning by recurrence* on the series of natural numbers. They show that the source of both calculating on the series of natural numbers (knowledge of algorithms) and inferring properties about its elements (knowledge of proof by induction) is inherent to the construction of the series of natural numbers (knowledge of inductive definitions). Regarding the meaning of induction and recursion in computer science and in mathematics, it can be said that this is a single concept nourished by knowledge on three areas: inductive definitions of structures, recursive algorithms defined on said structures and proofs by induction on the elements of those structures. We use the expression induction-recursion to mean that concept and our motivation follows from tenets of above cited work. In previous works [4, 3] we apply those principles to learn about the relationship between the learning of recursive algorithms on inductive structures different from natural numbers, and students' understanding of said structures. We have found important obstacles on the latter which play a central role in the learning of induction-recursion.

Accordingly to those results the goal of this work is to learn about the construction by the students of the concept of structure isomorphic to natural numbers, that is to say, generated

[1] The term structure has two different meanings in this paper: on the one hand it refers to the mental structures as defined by Piaget (schemes) and on the other hand it refers to inductive structures common in mathematics.

by inductive rules: there are initial elements (base case rule), other elements are generated by the application of constructive functions on previous elements (inductive rule) and these are the unique elements of the structure (clousure rule).

This kind of study is relevant for computer science education research because there is a broad consensus about the relevance of induction-recursion in computer science studies and, at the same time, it is considered both by students and by teachers, hard to learn and teach. Investigations about the learning and teaching of the concept of induction-recursion can be found at least since the 1970s, both in mathematics education and in computer science education based on theories as mental models, phenomenography and constructivism.

The term mental model is used by cognitive psychologists such as Johnson-Laird to define cognitive representations of knowledge. Karl Schwamb in "Mental Models: A Survey"(1990), indicates that mental models are subjects' representations of knowledge about particular situations or phenomena. In the case of learning recursion, several authors refer to mental models to describe the knowledge that the students acquire when introduced to the concept, in most of the cases using some programming language or environment. A mental model is said to be *viable* if it allows the students to accurately and consistently represent the mechanism of recursion and is said to be *non viable* if their representations show misconceptions. On the other hand, a *conceptual model* is designed by the teacher to teach the concept of recursion, while a *mental model* represents the understanding that the learner constructs. Within this theory, several misconceptions about the flow of control or the behavior of recursive procedures and its relationship to iteration are detected and classified in *active*, *step*, *syntactic or magic*, *copies* and *loop* mental models, both for the case of novice and expert students [12, 8, 7, 13].

In [2] the author follows a phenomenographic tradition of research developed in Sweden, based on exploring and describing the cognitive relations between individuals and the world. In the chapter about recursion, she describes that this topic is taught to students using the programming language SML. Then, the students are asked to solve some problems and answer some questions about their solutions and their works are analyzed and classified into three conceptions of recursion: as a construct in SML, as repetition and as self reference.

Constructivist researchers both in mathematics education and in computer science education often refer to the theory of Jean Piaget. For instance in [1] the author says "these concepts come from the seminal work of Jean Piaget" referring to knowledge construction (page 4). In [5] the learning of induction is described by mathematics education researches using Piaget's theory, genetic epistemology. Works like this have influenced our adoption of some general principles of that theory, regarding the construction of concepts and specific explanations about the process of generalizing assimilation [9, 10]. The main ideas are briefly described below.

- The processes involved in the construction of knowledge are associated with mental structures (schemes) which are generated as a result of, and operate with mental structures already formed, for example, in relation to a new problem.
- Instrumental knowledge: It is the knowledge constructed by subjects in the process which is activated when they attempt to solve a problem given by a specification.
- Conceptual knowledge or conceptualization: the construction of conceptual knowledge includes essentially two components: on the one hand, subjects become aware of their coordination of actions as well as the modifications of the objects. On the other hand, facing new problems requires transforming the cognitive resources to take into account variations and similarities. This transformation produces new mental structures (the assimilated concept).
- Comprehension: It is the result of applying conceptual knowledge to solve any problem of appropriate complexity.

The specific explanations taken into account to design this study are due to Cellérier's description in "Piaget Today" [6]. Cellérier explains the reciprocal assimilation of schemes which occurs with a new problem which is assimilated, first of all, to a familiar scheme. The novelty of the problem may or may not be an obstacle in the application (constructive generalization) of the scheme [10]. The obstacle may be represented as a new subproblem, which in turn is assimilated to the subscheme which may solve it, and thus the original scheme may be reapplied (until a new obstacle appears). There are no guarantees as to the solution of all obstacles (subproblems) by a subscheme. The effect of this cognitive strategy (unconscious to the individual) is to combine previous solutions in a new solution with its own function (process), resulting in the synthesis which is the actual core of constructivism. From the perspective of constructivism, structures are coordinated schemes, applicable to a wide range of problems, adapted to cooperate with other schemes within the internal epistemological universe. The concept of number is an example of a structure in this sense.

This article is divided into the following sections: section 2 includes the methodology of research; section 3 includes the development of the stages of the study; section 4 presents conclusions and further work and the list of references follows.

2 Methodology of research

The methodology of research is based on Cellérier's considerations and consists on investigating the pertinence of the following proposition: the knowledge that the students have constructed on the concept of the series of natural numbers can be generalized (in the sense of Cellérier) to a new inductive structure. Therefore, the students should be able to find out the defining rules the latter. To determine to what extent this proposition is correct, two instructional instances were conducted posing a problem about a structure isomorphous to natural numbers, in the sense that there are an initial element and a constructive function of an element to another one (successor).

The subproblems with combined solutions that can provide useful information have been identified as:

1. Identifying the first element.
2. Passing from one specific element to its successor.
3. Passing from a generic element to its successor.
4. Identifying the elements generated in this passage as the only elements of the structure.
5. Identifying the predecessor of a specific element (inverse to what is described in 2).
6. Identifying the predecessor of a generic element (inverse to what is described in 3).

Each subproblem is presented to the students as questions they must respond in writing. The purpose of these questions is to activate, in the students' thought, a generalization and specialization process of previous schemes, corresponding to the solutions of subproblems for the series of natural numbers. If one of the subproblems becomes an obstacle which cannot be assimilated to a subscheme, the students are presented with new questions or reformulated questions in order to direct their cognitive strategy towards overcoming the obstacle. The written answers are compared with the correct ones (provided by the investigators) to assess the difficulty level and to reformulate the question or to formulate a new one. We measure the progress in conceptualization as the distance of the students' answers with the correct answer.

The first and second stages were conducted with 8 and 7 voluntary students of Technical High School in Montevideo, aged 15 to 17. They had no previous formal instruction on induction or recursion which is an advantage to avoid preconceived ideas (often erroneous).

3 Development of the study

The study was conducted in two stages, where the second one arises from the analysis of the information gathered in the first one. In the first stage the students participated of three-hour meetings on two occasions during a period of forty five days. In the first meeting, the inductive definition of an infinite set is given by enumerating some elements and several questions are posed. In the second meeting, questions are reformulated or new questions are posed based on the analysis of the first answers. The goal was to encourage the students to find out the rules of the definition (base, inductive, clousure). The analysis of the information questioned the validity of supposing that the students comprehend the series of natural numbers as an inductive structure. Hence, a second stage of the study was designed and conducted to clarify the point. In the second stage seven students participated in one meeting of one hour and a half.

3.1 First stage

In the first stage, the definition of a set is presented in the problem below.

Problem: *In a faraway planet, there are some living creatures whose DNA is similar to ours, formed by chains of adenine A, thymine T, cytosine C and guanine G. We have little information on these creatures; we only know that they are very primitive, and that they divide in different and separate groups. After many explorations, robots collected DNA information of different groups of extra-terrestrial creatures, and sent it to earth hoping computer science experts would be able to establish the particularities in the DNA of the different groups of creatures from this faraway planet. The data sent by robots are DNA chains of different individuals of the group, a DNA chain being a succession of letters A, T, G, C, such as AGGCGGTAAT. We have received the DNA data of a subset of living creatures defined as:* **Group = {AAAGCTAAA, AGCTA, AAGCTAA, GCT, AAAAGCTAAAA, . . . }**. *This Group could have infinite creatures and thus we cannot indicate them all, but if we observe the known elements (such as the referred elements), we may see that these elements have been constructed or formed in a particular way, starting from an initial element.*

After the problem statement is given, questions from the three main following categories are asked:

- Questions 1 to 4, are related to the rules which define the set. The purpose of these questions is to have the student know that there is an initial element, GCT and that each element is generated from a predecessor by applying an inductive rule (to add an A on each side of the predecessor)
- Questions 5 to 7, are related to application of the rules. The purpose of these questions is, on the one hand, to help students understand the concept of closure, and on the other, to identify the relation between a generic element (different to the first element) and its predecessor (inverse to the inductive rule)
- Questions 8 to 10, related to the properties of the elements of the set. The purpose of these questions is to establish how much students rely on the series' regularity, for their understanding of proofs by induction. This category is included because the concept of induction has two aspects: on the one hand, the inductive definitions of sets and on the other hand the proofs by induction of properties of the elements of the set. As pointed out in [11] the source of reasoning by induction lies in the processes of constructing the inductive structures.

The following is the description of the questions of each category (designed considering the subproblems identified, described in the previous section) and the analysis of the students' answers for which the regulatory criteria, originated in the theoretical framework concepts, were used. Qi stands for question nr i. Recall that the questions of the second meeting were formulated after analyzing the responses of the first meeting.

3.2 Rules that define the set (questions 1 to 4)

This category of questions studies the conceptualization of the students of the initial element and of the relation of one element with the successor.

First meeting
Q1: Which one is the initial element for the Group?
Answer: . . . is the initial element for the Group.
Q2: If you could add only two letters and we provide you with the AGCTA element, how do you reach a new element in the Group set?
Q3: Given a generic element of the Group, is it possible to construct another element from it? In which way?

The purpose of the first question is to ask the students to identify the base case of the inductive structure. Although there is more than one correct answer, we expect that the students write GCT in the dotted points of question 1, because we believe that it is possible to generalize the knowledge of the series of natural numbers to this structure, with the mechanisms described by Cellérier [6], briefly mentioned in section 2. For the same reason, many questions are deliberately imprecise.

However, three students answered that the initial element is A, and the others answered that it is GCT. Regarding question 2, five students generated an element that DOES NOT belong to the Group. Incorrect answers included alteration of the order of letters, for example, stating that ACTAG is a new element constructed from another element. In Q3 it is expected that students who have found the answer of question 2 for a particular case, discover the rule for a generic case and succeed in its formulation. The study shows that this construction is not simple: only two students describe how to generate a new element correctly (adding an equal number of A's on both sides)i and all students used particular cases in their answer to Q3. These answers make us to believe that there is a visual factor involved. On the one hand, in Q1, since the elements of a set are read from left to right and A is the first letter that students read, and on the other hand, in Q2 and Q3, since the ellipsis in the definition of Group leads students to believe that they may generate new elements in any way.

In the **second meeting** Q1 and Q2 are reformulated for each student based on their previous answers and Q3 is repeated.

Second meeting
Q1: Note that ALL must be constructed from that initial element, then, given A as initial element, how do you construct GCT from A? Also note that A is not an element of the set.
Answer: Rule: . . . is the initial element for the Group.
Q2: Does the element you indicated in the answer to question 2 in the first meeting, belong to the Group? Answer again, noting that the new element constructed by adding letters must belong to the Group.
Q3: Given a generic element of the Group, is it possible to construct another element from it? In which way?

Note that those questions force students to pay attention to the form of the visible elements and induces the idea of elements following the same pattern, even if not present.

There is evidence of some progress between the first and second meeting, since, on the one hand the students answered correctly both questions 1 and 2, which shows that the idea of the rules has been introduced, not without difficulties, as revealed by the fact that only one student improved his answer to question 3 although still using a concrete case. Since this question refers to the concept of "generic element" we recall the study of Matalon, summarized in next subsection.

3.3 The generic element

In [11] Benjamin Matalon publishes a chapter entitled *Recherches sur le nombre quelconque*, in which he analyzes the relation between the generic element concept and the reasoning by induction, since such requires to prove that P(n) $\rightarrow$ P (n+1) for a generic number n and a given property P. Matalon works with the structure of natural numbers, stating that it is necessary to abstract all the particular properties in n, except the property of being a number, that is, an element belonging to the series of natural numbers. Matalon explains that Fermat made his arithmetic demonstrations using a particular number, but taken as a generic number, for example, the number 17. If none of the specific properties of the number 17 are involved in the demonstration, then the demonstration could be considered valid for all numbers. He adds that in geometry, when a property is to be proven and the statement is "given a generic triangle" a particular triangle is drawn, avoiding right triangles, equilateral triangles or isosceles triangles, and not involving particular properties of the triangle in the demonstration of the property. Among other things, Matalon concludes that to construct the concept of the "generic" element, it would be necessary to perform a generic action, that is, the repeated action to generate a generic element[2]. To extend this result, in the second meeting with students, they are asked to fill in a table which introduces repetition of the action that generates new elements from given ones, or which writes elements which are predecessors to given elements, as shown below:

Second meeting
Q4: Complete the table below and rewrite the set Group, ordering elements based on their length (number of letters). Then, fill in the dots:
Rule: Given a generic element . . . of the Group, then . . . is a new element of the Group.

Predecessor	New element
AGCTA	
	AAAAGCTAAAA
GCT	
	AAAGCTAAA
	AαA
α	

The rows left blank are aimed to induce the students in writing their own elements. All students were able to state correctly the rule in Q4 using α as the generic element. The study of the construction of a relation between one element and its successor, specially the evolution of the transmission between the repetition of actions and their iterative results, that is, between the action of the individual and the result which transforms the object, has been studied in [11]. Each action repeated after its predecessor differs from this in its range within the order of succession of actions and at the same time adds one element to the collection formed until the previous iteration. Once the subjects establish coordination between the succession of their actions and their results, a local synthesis specific for these actions is created, between the order of the succession of actions and the growing number of the collection of objects. This extends the construction of the structure with an aspect of recurrence reasoning, in which the most significant generalization is passing from one element to its successor. In this way, Piaget explains the construction of the series of natural numbers, by a synthesis between serialization and inclusion of classes [11]. For this case of construction of the inductive set, question 4 (where in each row of the table an action is performed and elements are ordered from shorter to longer), induces students to establish a similar synthesis, with which they may construct relations between one

[2] We recall that in genetic epistemology, the actions of the subject play a central role in generating knowledge.

element and its successor or between one element and its predecessor. The construction of said relations is the basis of inductive reasoning, both for the definition of inductive structures and recursive algorithms [3] as for proofs by induction.

We end this subsection with the answers used to compare students' responses to questions 1 to 4:
Q1: The initial element of the Group is GCT.
Q2: From AGCTA, AAGCTAA can be generated.
Q3: From a generic element α of the Group, AαA can be generated.
Q4: Group = {GCT, AGCTA, AAGCTAA, AAAGCTAAA, AAAAGCTAAAA, ...}
Given a generic element α of the Group, then AαA is a new element of the Group.

3.4 Applying rules (questions 5 to 7)

Once students have worked in the construction of answers of questions 1 to 4 giving rise to base and inductive rules, questions 5 to 7 are asked. The purpose is to draw the students' attention to the application of the **rules already defined by themselves**. Q5 is aimed to induce the students to realize the difference between those elements which may be constructed with the rules and those which may not. Q6 is aimed for the student to become aware that every chain different from GCT, has been generated from an element which is its predecessor.

First meeting
Q5: Given the element AGCT, may you construct a new one in the Group by applying your previous statements? Why or why not? Write down some chains which cannot be formed by applying the rules you stated.
Q6: Verify that all the chains from the Group have been constructed by applying the rules you wrote and underline the predecessor element in each case, when applicable.

All students pointed out that GCT (initial element) is a predecessor to all the other elements. This fact, confusing the initial element with the immediate predecessor is revealed as one of the most significant obstacles in the conceptualization of the structure. To help students overcome this obstacle, in the second meeting, after filling in the table (Q4), a new question about the structure of chains is asked (Q7 below) before going further into the predecessor issue by repeating question 6.

Second meeting
Q7: Given that all the chains of the Group have the structure you mentioned in Q1 and Q4, could there be a chain different to the initial one, which does not end with an A?
Could there be a chain with more than one C?
(New Q6): Verify that all the chains in the Group have been constructed by applying the rules you wrote in Q1 and Q4. Indicate, in each case, which is the predecessor element, when applicable. You may use the table.

We see progress of the students as to the first meeting, which would be explained by the use of the table. All students answered Q7 correctly, revealing some progress in conceptualization. However, answers to new Q6 show how hard is the construction of the relationship between an element and its predecessor. One of the students asked: Is GCT α? This means: Is GCT the predecessor of a generic element? This uncertainty of the student indicates an analysis of the relation of the predecessor with the current element. However, this student, and all students, answered that the initial element GCT is the predecessor to each element. This obstacle appears as one of the most important obstacles in the construction of the concept of induction, and thus further research is necessary to learn about its source. For example, which subscheme should this subproblem be assimilated to, and why it is not, and which other subproblems should be previously posed.

We end this subsection with the answers used to compare students' responses to questions 5 to 7:
Q5: No, I cannot because AGCT does not belong to Group.
Examples of chains that cannot be formed with the rules: AGCT, AAGCT, GCTA, AGCTAA.
Q6: Group = {GCT, AGCTA, AAGCTAA, AAAGCTAAA, AAAAGCTAAAA, ...}
Q7: No, all the chains different from GCT end with A. No, there cannot be more than one C in every chain.

3.5 Questions regarding properties (8 to 10)

The last group of questions (8 to 10) posed in the second meeting is related to the following property of the elements of the set: all elements have an equal number of A's on the left and on the right. One of the objectives is that students express correctly the statement of the property and the other one is that they express confidence about that all elements meet this property. Both constitute the basis to learn the method of proofs by induction of any property. The questions are the following:

Q8: Fill in the dots:
If a chain of the Group includes n A letters on the left of GCT, then the chain has a total of ... A letters.
If a chain in the Group includes a total of n A letters then it has ... A letters on the left of GCT.
Q9: Fill in: Let ... be a chain of the Group, then said chain has the property ...
(Write all the properties you believe apply).
Q10: Based on the previous information, could you say that all chains of the Group have the property ...?

All students use the variable n to answer correctly question 8. There were some events which prove some progress regarding understanding of the structure:

- All students but one mentioned that the property is to have an equal number of A's on the right and on the left and that the initial element is GCT
- Some students used the symbol α in the second subquestion of question 9. We transcribe one of the answers to highlight the progress made: "Given an α chain, it has the properties of being constructed from GCT and of having an equal number of A letters on both sides"
- All the students believe that every element of the set has the property (affirmative answers to question 10). This proves they rely on the regularity of the series of inductive chains, which for Piaget is a positive event in the study of the series of natural numbers [11].

We end this subsection with the answers used to compare students' responses to questions 8 to 10:
Q8: The total number of A in a chain with n A letters is 2*n. If the total number of A letters is n, there are n/2 A letters on the left of GCT.
Q9: Let α be a chain then either α is GCT or α has equal number of A letters to the left and to the right of GCT.
Q10: All the chains satisfy the property of Q9.

3.6 Summary of results of the first stage

In this section we summarize the answers of students to the questions and the main problems that need further investigation. The answers of the students to the questions reveal fundamentally three main facts:

- In the answers to question 2 of the first meeting most of students included in Group, elements that do not follow the pattern of the visible elements.

- The reformulation of questions of the first meeting posed in the second meeting seems to help the students to correctly answer Q1 and Q4. That means that they succeed in stating the base and inductive rules of the definition of the structure. The correct answers to Q2 and Q7 (second meeting) reveal advances in the conceptualization of the closure rule, as well.
- However, the students did not succeed to overcome the confusion between the initial element of the Group (GCT) and the predecessor of each element: in the answers to Q6 (both meetings) all students pointed out GCT as the predecessor of each element. Our interpretation is that, despite the correct definition of the rules, the concept of inductive structure is not attained, that is to say the corresponding mental scheme has not been constructed.

These considerations have provided insight on some of the problems which need further investigation. To begin with, we recall our proposition which is that knowledge about the series of natural numbers may help in constructing knowledge about other inductive structures, based on Cellérier's work. Our start point is then that the students already have conceptualized the series of natural numbers because they work and succeed on solving problems from early education. Why do they fail in generalization and specialization the schemes of the series of natural numbers to construct knowledge about the structure of the problem posed in this study? One of the possible answers is that our premise is wrong: the students have not constructed *conceptual knowledge* about the series of natural numbers, despite the *instrumental knowledge* that they reveal in solving problems.

This new perspective leads us to pose the following question: do students have similar difficulties if problems are about natural numbers? Depending on the answer, we can obtain information about whether the obstacle is the original scheme or the process of its generalization.

In order to find an answer, we carried out a second stage of this study in which the students were asked to solve two problems with six questions each, during a meeting of an hour and a half. A brief description of the problems is included in next section.

3.7 Second stage of the study

The main goal of the second stage of the study is to learn about whether difficulties similar to those already detected appear when the students work with inductive structures of natural numbers. The base, inductive and closure rules of the definition of two subsets of natural numbers are given. Several questions are posed to encourage students to recognizing the initial elements as the only ones that have no predecessor, to generating new elements from previous ones and to pointing out the predecessor of any element.

The questions are similar in both problems, as shown below, and the students must respond in writing.

Problem 1: Generating consecutive even natural numbers.

A set A of natural numbers is generated by the following rules:

- Rule 1: $4 \in \mathrm{A}$
- Rule 2: if $\alpha \in \mathrm{A}$ then $\alpha + 2 \in \mathrm{A}$
- Rule 3: No other numbers are included in A.

Answer the following questions:

- Q1: Write down some elements of A.
- Q2: Is 67 an element of A? Why or why not?
- Q3: If $36 \in \mathrm{A}$, which is the predecessor of 36?
- Q4: How do you find it?
- Q5: If x is an element of A different from 4, which is the predecessor of x?
- Q6: Complete the table below (all the elements belong to A).

Predecessor	New element
6	
	24
58	
	316
	x
x	

- Q7: Is there any element in the set A with no predecessor? Which one?

Problem 2: Generating non-consecutive odd natural numbers.

A set B of natural numbers is generated by the following rules:

- Rule 1: 7 ∈ B
- Rule 2: if $\alpha \in$ B then $2 * \alpha + 1 \in$ B
- Rule 3: No other numbers are included in B.

Answer the following questions:

- Q1: Write down some elements of B.
- Q2: Is 11 an element of B? Why or why not?
- Q3: If 63 ∈ B, which is the predecessor of 63?
- Q4: How do you find it?
- Q5: If x is an element of B different from 7, which is the predecessor of x?
- Q6: Complete the table below (all the elements belong to B).

Predecessor	New element
15	
	63
7	
	255
	x
x	

- Q7: Is there any element in B with no predecessor? Which one?

In the responses the following facts are detected:

- Regarding the initial element (questions Q1, Q6 and Q7).
  - In the answers to the first question of both problems, elements not belonging to the sets were included, for instance, 2 for the case of A and 1, 3 for the case of B (recall that initial elements are 4 and 7 respectively).
  - The empty rows of the table were filled out with wrong elements.
  - One student answered question 7 of problem 1 correctly, while the other responses to that question were wrong. For instance, many answered that the element with no predecessor is 0 or 2 for the case of set A.
- Regarding predecessor of concrete elements (Q2, Q3, Q4, Q6).
  - Question 2 was correctly answered in problem 1, but incorrectly in problem 2. For instance, some students said that 11 belongs to B because 11 = 5 * 2 + 1, without noting that 5 does not belong to B.
  - In question 3 one student answered that the predecessor of 63 is 61 and three students gave not answer at all. Three students gave the correct answer for questions 3 and 4.
- Regarding predecessor of generic elements (Q5, Q6).

- One student answered that 4 is the predecessor of x in problem 1 and gave no answer for this question in problem 2. Two students gave correct answers in both problems, but one of them filled the table (Q6) incorrectly. The remainder of students answered the question just for the first problem.

It was observed that in the second problem several questions have been left unanswered and there were found more errors than in the first. The first problem is simpler than the second one in the sense that the inductive rule involves just the addition while in the second two operations are involved: multiplication and substraction.

We end this subsection with the answers used to compare students' responses to the questions of the problem 1 and of the problem 2:

Table 1. Answers used to compare students' responses

Q	**Subset A**	**Subset B**
1	4 6 8 10	7, 15, 31
2	No, all are formed adding 2 to an element, starting in 4	No, because 5 does not belong to B.
3,4	$34 = 36 - 2$	$31 = \frac{63-1}{2}$
5	$x + 2$	$\frac{x-1}{2}$
6	table of elements of A (see below)	table of elements of B (see below)
7	4	7

Table 2. Tables of Q6 for subsets A and B

Predecessor	New element	Predecessor	New element
6	8	15	31
22	24	31	63
58	60	7	15
314	316	127	255
10	12	255	511
$x - 2$	x	$\frac{x-1}{2}$	x
x	$x + 2$	x	$2 * x + 1$

4 Conclusions and Further Work

There follows a classification of some of the types of errors appearing during the first stage of the study and a summary of them related the regulatory criteria from the theory (table below).

- Type 1: There is an error influenced by a visual factor (defining A as the initial element)
- Type 2: There are two types of type 2 errors: a) given an element of the Group, construct another one which IS NOT part of the Group (for example from AGCTA to AAGCT) and b) pass from an element which is NOT from the Group to a new element (for example from AGCT to AGCTA). (In the table below, x and x + 1 are used to denote an element and its successor respectively.)
- Type 3: Using particular cases for the generic element
- Type 4: Confusing the predecessor with the initial element
- Type 5: An influence of preconceived ideas (confusing property with element, confusing language with metalanguage)

Table 3. Summary of types of errors of the first stage

Question	Objective	Regulatory criteria	Errors
1 to 4	Identifying the first element	Local synthesis/serialization	Type 1
1 to 4	Passing from x to x+1	Local synthesis/serialization	Type 2
1 to 4	Clousure	Regularity of the series	Type 2
1 to 4	The generic element	Repetition of the action	Type 3
5 to 7	Application of the rules	Constructive generalization	Type 2
5 to 7	Passing from x+1 to x	Local synthesis/serialization	Type 4
8 to 10	Thinking properties	Regularity of the series	Type 5

The different types of errors are related. In general terms, it could be said that one type of error leads to other types. To set an example, students who cannot understand the relation between one element and its successor (or predecessor), add to the set, elements which do not belong there. Confusing the initial element of the structure and the predecessor to a generic element is an obstacle both for the proof by induction – since one element (different to the initial element) satisfies a specific property because the predecessor does – and for the definition of recursive algorithms – where the result for each element is constructed using the result for the previous element–.

Although the information gathered in the second stage has to be more deeply analyzed, it can be said that the same types of errors have been detected in both stages of the study. We believe that the facts pointed out in previous section, show evidence that the obstacles partially lie in students' lack of *conceptual knowledge* of the series of natural numbers as an inductive structure, despite they are at pre-university level. We believe that this affects the learning on induction-recursion and that it is necessary to help students to construct *conceptual knowledge* on the series of natural numbers from their *instrumental knowledge*. The objectives of our next study shall focus on that issue.

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